3.7 Rates of Change in the Natural and Social Sciences

In this section we will use the idea of the derivative in applications for the fields of physics, chemistry, biology, economics, and other sciences.

So far we have seen that the derivative of y with respect to x can be interpreted as the change of y with respect to x, or that the rate of change from x_1 to x_2 and the corresponding change in y is found by the difference quotient:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$\frac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to x over the interval $[x_1, x_2]$. It's limit as $\Delta x \to 0$ is the derivative of $f'(x)$, which is the instantaneous rate of change of y with respect to x.

PHYSICS

We have already said that if $s = f(t)$ is the position of a particle, then,

$v = f'(t)$ is the velocity of a particle, and

$a = v'(t) = f''(t)$ is the acceleration.

Example: The height (in meters) of a projectile shot vertically upward from a point 2 meters above ground level with an initial velocity of 24.5 m/s is $\boldsymbol{h = 2 + 24.5t - 4.9t^2}$ after t seconds. (This formula could also be written as $\boldsymbol{h = -4.9t^2 + 24.5t + 2}$)

a) Find the velocity after 2 seconds.

b) When (at what value of t) does the projectile reach its maximum height?

c) What is the maximum height?

a) To find velocity at 2 seconds, find $h'(2)$. $\quad h' = 24.5 - 9.8t \quad h'(2) = 24.5 - 9.8(2) = \boldsymbol{4.9m/s}$

b) Notice that the maximum height occurs when $h(t)$ has a horizontal tangent, which is when $h'(t) = 0$. So if we solve for the t value that gives us $h'(t) = 0$, then this is the t when the object has max. height.

$h'(t) = 0 \;\rightarrow\; 24.5 - 9.8t = 0 \;\rightarrow\; 24.5 = 9.8t \;\rightarrow\; t = \frac{24.5}{9.8} \;\rightarrow\; \boldsymbol{t = 2.5\ seconds}$ Translation: 2.5 seconds after the object was shot vertically upward it was at it's maximum height.

c) The maximum height is $h(2.5)$

$h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 2 + 61.25 - 30.625 = \boldsymbol{32.625\ meters}$ (maximum height)

ECONOMICS

As you might have figured, there are also ways to write business problems using mathematical models. The revenue, cost, and profit functions can all be found using math. In addition, we can also find their derivative, the derivatives of such functions are called **marginal functions.**

Example: Suppose that ***C(x)*** is the total cost that a company incurs in producing ***x*** units of a certain commodity. The **marginal cost** is given by ***C'(x)***, where ***C'(x)*** is the instantaneous rate of change of the cost with respect ***x***, the number of items produced.

Example: Suppose the cost (in dollars) for a company to produce ***x*** pairs of a new line of jeans is:

$$\boldsymbol{C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3}$$

a) Find the marginal cost function.

b) Find ***C'(100)*** and explain its meaning.

a) $C'(x) = 3 + 2(0.01)x + 3(0.0002)x^2 \rightarrow$ $\boldsymbol{C'(x) = 3 + 0.02x + 0.0006x^2}$

b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$\mathbf{11}$ **\$11/pair of jeans. The rate at which the cost is changing for the 100th pair of jeans produced.**

Let ***R(x)*** be the revenue from selling ***x*** items at a certain price. If the cost is given by ***C(x)***, then the profit, ***P(x)*** is as follows: ***P(x) = R(x) – C(x).***

The ***break-even*** point occurs when ***P(x) = 0***. Which is where ***R(x) = C(x)***.

Example: If $C(x) = 100{,}000 + 160x - 0.2x^2$ and $R(x) = 800x$, find the following:

a) Find ***P(x)***

b) How many items must this company produce to make a profit?

a) $P(x) = R(x) - C(x) \rightarrow P(x) = 800x - (100{,}000 + 160x - 0.2x^2)$

$$= 800x - 100{,}000 - 160x + 0.2x^2 \text{ (simplify)}$$

$$= 640x - 100{,}000 + 0.2x^2$$

b) Set $P(x) = 0$ to find the break-even point.

$0.2x^2 + 640x - 100{,}000 = 0$ (Use the quadratic formula to solve for ***x***.)

X = 149.5 or – 3,347.3 (Making a negative number of items does not make sense so we choose the positive value for x. To make a profit, this company must sell at least **150 items.**